EFFECT OF A TEMPERATURE GRADIENT ON THE HEAT TRANSFER DURING LAMINAR NATURAL CONVECTION ALONG A VERTICAL WALL

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UDC 536.25

Results are shown of calculations and of an experimental study concerning the heat transfer during laminar natural convection along a vertical wall, in the general case when the temperature excess varies along the wall height according to an arbitrary law.

The effect of an anisothermal wall on the heat transfer coefficient during natural convection was most thoroughly analyzed in [1]. A solution to the problem was sought through a similarity transformation of a system of partial differential equations, this system including the mass equation, the momentum equation, and the energy equation. The resulting system of ordinary differential equations was then integrated numerically for two special cases, namely for a power-law and for an exponential-law distribution of the temperature excess. Integral relations for the boundary layer were used in a later study [2]. The profiles of temperature and velocity were represented by fifth-degree polynomials. An approximate solution was obtained for a power-law temperature distribution.

In order to solve the problem for the general case of an arbitrary temperature distribution along the height of a vertical wall, here the authors will use the integral equation of heat transfer in a boundary layer

$$\frac{d\delta_T^{**}}{dx} + \left(\frac{1}{v_x^0}\cdot\frac{dv_x^0}{dx} + \frac{1}{\theta_w}\cdot\frac{d\theta_w}{dx}\right)\delta_T^{**} = \frac{\alpha_L}{\rho c_p v_x^0}, \tag{1}$$

where $\delta_T^{**} = \int\limits_0^\delta (v_x/v_x^0)(1-(t_w-t)/(t_w-t_f))dy$ is the heat-content thickness. Equation (1) has been derived as in the case of forced convection, with the maximum velocity in the boundary layer v_x^0 replacing the main-stream velocity.

The magnitude of v_x^0 is determined from the approximate solution to the momentum equation

$$\frac{d}{dx}\left(\int\limits_0^\delta v_x^2\,dy\right) = g\beta\int\limits_0^\delta \theta dy - \nu\left(\frac{dv_x}{dy}\right)_w, \tag{2}$$

with the velocity profile and the temperature profile in the laminar boundary layer represented as

$$v_x = \frac{g\beta\theta_w\delta^2}{4\nu}\left[\frac{y}{\delta} - 2\left(\frac{y}{\delta}\right)^2 + \left(\frac{y}{\delta}\right)^3\right], \quad \theta = \theta_w\left(1-\frac{y}{\delta}\right)^2, \tag{3}$$

and assumed to satisfy the following boundary conditions:

$$y = 0 \quad v_x = 0, \quad \frac{\partial^2 v_x}{\partial y^2} = -\frac{g\beta\theta_w}{\nu}, \quad \theta = \theta_w,$$

V. I. Lenin Polytechnic Institute, Khar'kov. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 26, No. 3, pp. 411-419, March, 1974. Original article submitted July 31, 1973.

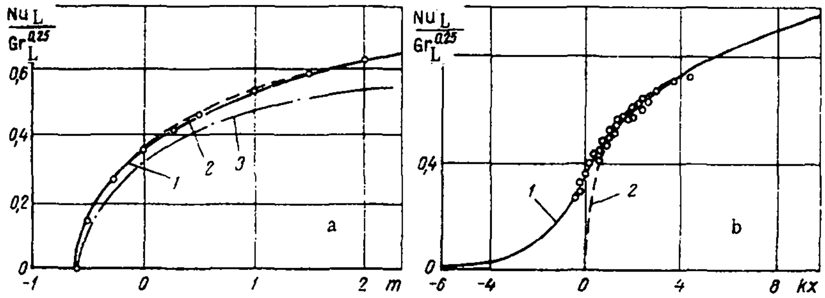

Fig. 1. Comparison between test data and theoretical relations for various profiles of the temperature excess: (a) power-law profile, (b) exponential-law profile; 1) according to formulas (16) and (14); 2) numerical solution in [1]; 3) approximate solution in [2]; dots represent test values.

$$y=\delta=\delta_T \quad v_x=0, \quad \frac{\partial v_x}{\partial y}=0, \quad \theta=0, \quad \frac{\partial\theta}{\partial y}=0.$$

The solution to Eq. (2) will be

$$\delta=\left(\frac{112\nu^2}{g\beta}\right)^{0.25}\theta_w^{-0.4}\left(\int_0^x\theta_w^{0.6}\,dx\right)^{0.25},$$

$$v_x^0=0.393\left(\frac{g\beta}{\nu}\right)^{0.5}\theta_w^{0.2}\left(\int_0^x\theta_w^{0.6}\,dx\right)^{0.5}. \tag{4}$$

Then, in order to solve Eq. (1), it is necessary to express the law governing the heat transfer in terms of a relation between the heat transfer coefficient and the Reynolds number, both referred to the heat-content thickness. The sought relation can, to the first approximation, be found on the assumption that it does not depend on the altitudinal temperature gradient and that, consequently, it is identical to the law governing the heat transfer at an isothermal wall. An analogous assumption is usually made in the analysis of forced convection [3]. For θ_W = const let us now assume the power-law approximation

$$\mathrm{Nu_L}=\varphi\,\mathrm{Gr_L^n}, \quad \varphi=\varphi(\mathrm{Pr}).$$

Letting

$$\frac{d\theta_w}{dx}=0,$$

in (1), we obtain

$$\frac{d\delta_{T_0}^{**}}{dx}+\frac{\delta_{T_0}^{**}}{2x}=\frac{\varphi a x^{3n-0.5}\left(\frac{g\beta\theta_w}{\nu^2}\right)^n}{0.393\,\nu^{0.5}}. \tag{5}$$

The integral of Eq. (5) is

$$\delta_{T_0}^{**}=\frac{\varphi a\,\mathrm{Gr_L^n}}{3nv_{x_0}^0}.$$

From here

$$\mathrm{Re}_{T_0}^{**}=\frac{\delta_{T_0}^{**}v_{x_0}^0}{\nu}=\frac{\varphi\,\mathrm{Gr_L^n}}{3n\,\mathrm{Pr}}. \tag{6}$$

The maximum velocity in the boundary layer will now be expressed as a function of $\mathrm{Re}_{T_0}^{**}$:

$$v_{x_0}^0=0.393\left(\frac{g\beta\theta_w}{\nu^2}\right)^{0.5}\nu^{0.5}\cdot x^{0.5}=0.393\,\frac{\nu^{0.5}}{x}\left(\frac{3n\,\mathrm{Pr}\,\mathrm{Re}_{T_0}^{**}}{\varphi}\right)^{\frac{1}{2n}}. \tag{7}$$

After eliminating $\mathrm{Gr_L}$ and $v_{x_0}^0$ from the equality

$$\frac{\alpha_L}{\rho c_p v_{x_0}^0}=\frac{\varphi\,\mathrm{Gr_L^n}\lambda}{x\rho c_p v_{x_0}^0},$$

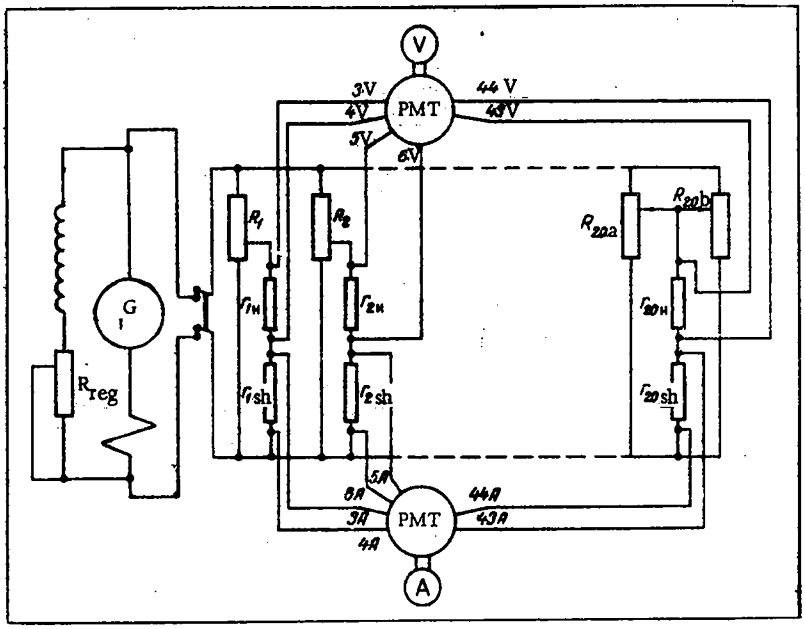

Fig. 2. Electric circuit of the heater system for the plate-calorimeter.

we have the law of the heat transfer

$$\frac{\alpha_L}{\rho c_p v^0_{x_*}} = A\,(\mathrm{Re}^{**}_{T_*})^{\frac{2n-1}{2n}}, \quad A = 2.544\,\nu^{0.5}\,(3n)^{\frac{2n-1}{2n}}\left(\frac{\varphi}{\mathrm{Pr}}\right)^{\frac{1}{2n}}. \tag{8}$$

By virtue of the earlier assumption that the law of the heat transfer is conservative with respect to the gradient of the temperature excess θ_W, we have

$$\frac{\alpha_L}{\rho c_p v^0_x} = A\,(\mathrm{Re}^{**}_T)^{\frac{2n-1}{2n}}, \tag{9}$$

where

$$\mathrm{Re}^{**}_T = \frac{\delta^{**}_T v^0_x}{\nu}.$$

When θ_W = var, then

$$\frac{d\delta^{**}_T}{dx} + \left(\frac{1}{v^0_x}\frac{dv^0_x}{dx} + \frac{1}{\theta_w}\frac{d\theta_w}{dx}\right)\delta^{**}_T = A\,(\mathrm{Re}^{**}_T)^{\frac{2n-1}{2n}}. \tag{10}$$

The solution to Eq. (10), with expression (4) taken into account, is

$$\delta^{**}_T v^0_x \theta_w = \left(\frac{A}{2n\nu^{\frac{2n-1}{n}}}\int_0^x v^0_x\,\theta_w^{\frac{1}{2n}}\,dx\right)^{2n}. \tag{11}$$

It follows from (1) that

$$\frac{d}{dx}(\delta^{**}_T v^0_x \theta_w) = \frac{q}{\rho c_p} = \frac{\mathrm{Nu}_L\,\theta_w\,\nu}{x\,\mathrm{Pr}}. \tag{12}$$

Combining (11) and (12) yields

$$\mathrm{Nu}_L = 1.5^{2n-1}\,\varphi\,\mathrm{Gr}^n_L\,\theta_w^{\frac{1-1.6n-2n^2}{2n}}\,x^{1-3n}\,\frac{\left(\int_0^x \theta_w^{0.6}\,dx\right)^{0.5}}{\left[\int_0^x \theta_w^{\frac{1+0.4n}{2n}}\left(\int_0^x \theta_w^{0.6}\,d\zeta\right)^{0.5} dx\right]^{0.5}}. \tag{13}$$

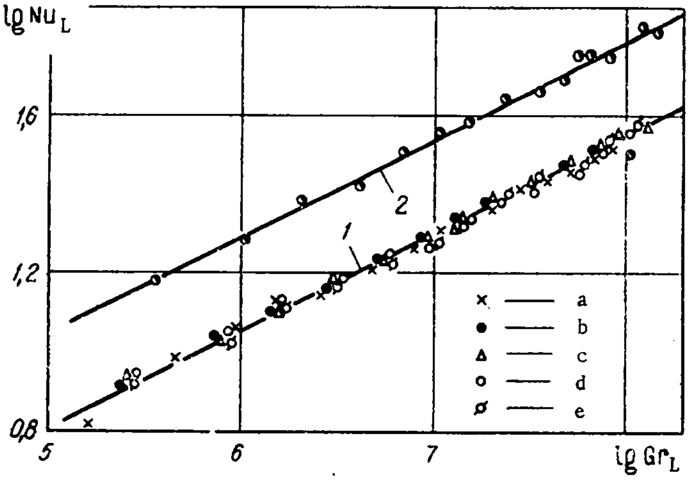

Fig. 3. Heat transfer (1) at an isothermal plate, (2) at a plate with a square-law temperature profile along its height: a) $\theta_w = 25°C$, b) $\theta_w = 50°C$, c) $\theta_w = 75°C$, d) $\theta_w = 100°C$, e) $\theta_w = 125°C$.

In the case of laminar flow we have n = 0.25 and, therefore,

$$\mathrm{Nu}_L = 0.818\,\varphi\,\mathrm{Gr}_L^{0.25}\,\theta_w^{0.95}\,x^{0.25}\,\frac{\left(\int_0^x \theta_w^{0.6}\,dx\right)^{0.5}}{\left[\int_0^x \theta_w^{2.2}\left(\int_0^x \theta_w^{0.6}\,d\xi\right)^{0.5}dx\right]^{0.5}}. \tag{14}$$

If for θ_w = const we assume that $\mathrm{Nu}_L = C(\mathrm{Pr\,Gr}_L)^n$, then relation (14) becomes

$$\mathrm{Nu}_L = 0.818\,C\,(\mathrm{Pr\,Gr}_L)^{0.25}\,\theta_w^{0.95}\,x^{0.25}\,\frac{\left(\int_0^x \theta_w^{0.6}\,dx\right)^{0.5}}{\left[\int_0^x \theta_w^{2.2}\left(\int_0^x \theta_w^{0.6}\,d\xi\right)^{0.5}dx\right]^{0.5}}. \tag{14a}$$

The mean heat transfer coefficient will be defined as the ratio of the mean-integral thermal flux density to the mean-integral temperature excess

$$\alpha_m = \frac{\int_0^l \alpha_L\,\theta_w\,dx}{\int_0^l \theta_w\,dx}.$$

Inserting α_L from the solution to (14), we obtain

$$\mathrm{Nu} = 1.636\,\varphi\,\mathrm{Gr}^{0.25}\,l^{0.5}\,\frac{\left[\int_0^l \theta_w^{2.2}\left(\int_0^x \theta_w^{0.6}\,d\xi\right)^{0.5}dx\right]^{0.5}}{\left(\int_0^l \theta_w\,dx\right)^{1.25}}, \tag{15}$$

where

$$\mathrm{Gr} = \frac{g\beta\theta_m\,l^3}{\nu^2}, \quad \theta_m = \frac{1}{l}\int_0^l \theta_w\,dx.$$

In the special case of a power-law temperature profile, when $\theta_w = dx^m$, the theoretical value of the local Nusselt number is

$$\mathrm{Nu}_L = 0.818\varphi\sqrt{\frac{2.5m+1.5}{\sqrt{0.6m+1}}}\,\mathrm{Gr}_L^{0.25}. \tag{16}$$

TABLE 1. Test Data on the Heat Transfer at a Plate with $\theta_W = dx^{-0.6}$

N	x, m	t_w, °C	θ_w, °C	I, A	V, V	α, W/m²·°C
1	0,008	95,6	81,6	3,28	0,332	—
2	0,024	63,4	49,4	1,97	0,192	0,513
3	0,040	45,9	31,9	1,34	0,163	0,386
4	0,056	39,5	25,3	1,16	0,144	0,428
5	0,072	35,7	21,7	0,97	0,135	0,251
6	0,088	33,3	19,3	0,92	0,128	0,294
7	0,104	31,3	17,3	0,88	0,123	0,366
8	0,120	30,1	16,1	0,867	0,120	0,445
9	0,136	28,8	14,8	0,840	0,113	0,438
10	0,152	27,8	13,8	0,828	0,110	0,51
11	0,168	27,0	13,0	0,80	0,109	0,551
12	0,184	26,4	12,4	0,694	0,094	0,075
13	0,200	25,7	11,7	0,65	0,088	—0,042
14	0,216	25,2	11,2	0,647	0,087	0,005
15	0,232	24,7	10,7	0,634	0,080	—0,083
16	0,248	24,3	10,3	0,640	0,086	0,118
17	0,264	24,0	10,0	0,614	0,082	0,027
18	0,280	23,7	9,7	0,609	0,0806	0,022
19	0,296	23,5	9,5	0,600	0,080	0,035
20	0,312	23,2	9,2	0,68	0,092	—

The solutions according to [1, 2] and the criterial relation (16) for Pr = 0.7 are compared in Fig. 1. The approximate solution (16) is almost identical to the numerical solution in [1] over the entire range of m from −0.6 to 3.0, for which Nu_L has been determined in [1]. We note that, with a temperature profile $\theta_W = dx^{-0.6}$, Nu_L is equal to zero on the entire wall surface.

The criterial relation (14) is valid for any temperature profile along the wall surface under which convection remains laminar. This approximate solution has been checked against test data for special cases of a power-law ($\theta_W = dx^m$) and an exponential-law ($\theta_W = be^{kx}$) altitudinal profile of the temperature excess.

The process of natural convection was studied in an unbounded air space. The local heat transfer coefficients were measured by the calorimetric method: a plate acting as the calorimeter was heated with transverse strips of alloy ÉI-442M foil 0.1 mm thick, this alloy having an electrical resistivity only very slightly dependent on the temperature. Each strip was 100 mm long and 15 mm wide, with a clearance of 1 mm between them. The plate, made of glass-Textolite 320 mm long and 4 mm thick, was placed in the vertical position. Both sides of the plate were covered with 20 such electric heater strips carrying dc current from a motor-generator set as shown in Fig. 2. Each matching pair of heaters on both sides at the same distance from the lower edge of the plate was connected in series and the power in each could be regulated independently through rheostats. This made it possible to establish a temperature field in the plate symmetrical with respect to its median section and, at the same time, also made it possible to regulate the temperature field along the height of the plate. The heat conduction through glass-Textolite was negligible: even during the most drastic temperature drop along the surface at $\theta_W \sim x^2$, the average heat leakage from one segment of the plate-calorimeter amounted to less than 1% of the heat generated by its electric heater.

The foil strips were capacitor-welded to copper shunts. One shunt of each strip was firmly fastened to the plate, but the other shunt was spring-mounted to allow for thermal expansion during heating. The foil strips were attached to the plate, since the spring tension ensured a sufficiently solid contact.

The local heat transfer coefficient was calculated according to the formula

$$\alpha_L = \frac{IV - Q_r}{F\theta_w}.$$

The current I and the voltage V were measured with model M1107 class 0.2 voltammeters. The temperature of the heat transfer surface and the ambient temperature were measured with calibrated Chromel −Copel thermocouples 0.2 mm in diameter and with a model R-307 class 0.015 potentiometer. The thermocouples, both the main and the auxiliary ones, were welded to the inner surface of the heaters, then laid along grooves specially cut in the plate, and finally brought out braided inside metallic shielding sleeves. The readings of the main thermocouples were used in the formulas for the heat transfer coefficients, while the readings of the auxiliary thermocouples served to estimate the heat leakage along each strip.

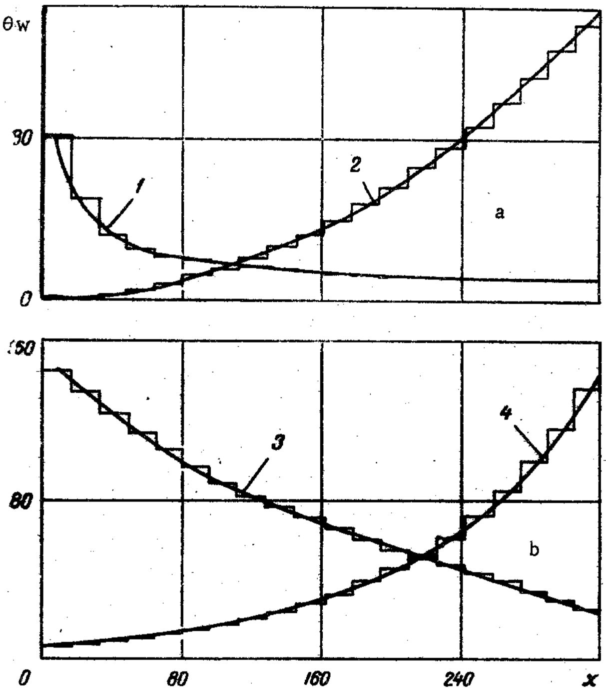

Fig. 4. Calculated and measured temperature profile θ_W (°C) along the plate height x (mm): (a) according to a power law with 1) m = −0.6, 2) m = 2.0; (b) according to an exponential law with 3) k = −5, 4) k = 10.

The first test series was performed at a constant temperature excess of 25, 50, 75, 100, and 125°C, for the purpose of checking out the test procedure. The results of these tests are shown in Fig. 3 (curve 1). The test values of the local Nusselt number Nu_L agree almost exactly with those according to the well-known theoretical formula in [4]

$$Nu_L = 0.36\, Gr_L^{0.25}, \tag{17}$$

which has been repeatedly confirmed by tests.

The effect of the temperature gradient on the heat transfer coefficient was studied in the case of a power-law profile of the temperature excess, with the power exponent equal to 2.0, 1.5, 1.0, 0.5, 0.25, −0.25, −0.5, and −0.6 respectively. The test data were then evaluated in terms of power monomials

$$Nu_L = C_m\, Gr_L^{0.25}.$$

The averaging values of C_m for each test series are compared in Fig. 1 with theoretical values. The empirical relation $Nu_L = f(Gr_L)$ corresponding to a square-law profile of the temperature excess is shown in Fig. 3 (curve 2). As compared to an isothermal plate, the heat transfer rate is in this case 70% higher. As the power exponent decreases, the heat transfer coefficient also decreases and, according to computed data, becomes zero when m = −0.6. The test data for this specific case are given in Table 1.

At x > 0.18 m the measured local heat transfer coefficients were by approximately two orders of magnitude smaller than according to formula (17) for an isothermal plate. At the lower edge the heat transfer rate was reduced less appreciably. At x = 0.024 m, for example, the local heat transfer coefficient was only 5.5 times smaller. An explanation for this may be, most probably, that the temperature profile along the height of the calorimeter plate could be regulated only discretely. On account of this, the profile of the temperature excess was reconstructed approximately only, in the form of a piecewise constant function deviating from the stipulated profile most markedly at low values of the argument (Fig. 4, curve 1). The overall thermal flux density, calculated on the basis of the mean heat transfer coefficient for an isothermal plate with a mean-integral temperature excess of 20°C, was found to be 13 times higher than the mean thermal flux density measured at $\theta_W = dx^{-0.6}$.

Tests were also performed with an exponential-law profile of the temperature excess, $\theta_w = be^{kx}$. The values of parameter k were made equal to 15, 10, 5, and −5 respectively. The profile of the temperature excess along the plate height is shown in Fig. 4b for k = 10 and −5 respectively. The test results are shown in Fig. 1 in terms of the relation $Nu_L/Gr_L^{0.25} = f(kx)$.

The theoretical relation shown in [1] (for Pr = 0.7) agrees with the test data only within kx > 0.5. This is, apparently, due to the singularity at k = 0 in the similarity transformation used in [1]. The values of local heat transfer coefficients calculated according to formula (14) agree with the test data over the entire test range of kx (see Fig. 1).

Thus, the approximate relation (14) obtained for the general case of an arbitrary temperature profile along the plate height has been checked against test data, with the profile of the temperature excess assumed to follow two different laws respectively and with various values assumed for k and m determining the rate of increase or decrease of the temperature excess in the direction of flow. Such an agreement between results of measurement and calculation also confirms, indirectly, the validity of the assumption that the law governing the heat transfer is self-adjoint with respect to the temperature gradient.

NOTATION

δ_T^{**} is the heat-content thickness;
δ is the thickness of the hydrodynamic boundary layer;
δ_T is the thickness of the thermal boundary layer;
x is the abscissas measured from the lower edge along the plate;
y is the ordinates along a normal to the plate;
l is the plate length;
α is the heat transfer coefficient;
λ is the thermal conductivity;
c_p is the specific heat;
ρ is the density;
ν is the kinematic viscosity;
$\theta = t - t_f$ is the temperature excess in the boundary layer;
$\theta_w = t_w - t_f$ is the temperature excess at the wall (plate);
t_w is the temperature at the plate surface;
t_f is the ambient temperature beyond the boundary layer;
t is the temperature in the boundary layer;
Nu is the Nusselt number;
Pr is the Prandtl number;
Gr is the Grashof number;
Q_r is the heat loss due to radiation;
F is the surface area of an electric heater.

Subscripts

L relates to local values;
0 relates to flow without temperature gradient.

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